

# BRST FORMALISM AND ZERO LOCUS REDUCTION

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**ABSTRACT.** In the BRST quantization of gauge theories, the zero locus  $\mathcal{Z}_Q$  of the BRST differential  $Q$  carries an (anti)bracket whose parity is opposite to that of the fundamental bracket. Observables of the BRST theory are in a 1 : 1 correspondence with Casimir functions of the bracket on  $\mathcal{Z}_Q$ . For any constrained dynamical system with the phase space  $\mathcal{N}_0$  and the constraint surface  $\Sigma$ , we prove its equivalence to the constrained system on the BFV-extended phase space with the constraint surface given by  $\mathcal{Z}_Q$ . Reduction to the zero locus of the differential gives rise to relations between bracket operations and differentials arising in different complexes (the Gerstenhaber, Schouten, Berezin–Kirillov, and Sklyanin brackets); the equation ensuring the existence of a nilpotent vector field on the reduced manifold can be the classical Yang–Baxter equation. We also generalize our constructions to the bi-QP-manifolds which from the BRST theory viewpoint correspond to the BRST-anti-BRST-symmetric quantization.

## 1. INTRODUCTION

The “BRST” quantization of general gauge theories in the Hamiltonian and Lagrangian formalisms includes the Batalin–Fradkin–Vilkovisky (BFV) [1] and Batalin–Vilkovisky (BV) [2] formalisms. From a geometric standpoint, these quantization formalisms deal with an even or odd QP manifold  $\mathcal{N}$  [3, 4], i.e., a symplectic or antisymplectic manifold equipped with a compatible odd vector field  $Q$  such that  $Q^2 = 0$ . This condition is ensured by imposing the *master equation* on the Hamiltonian function of the vector field  $Q$ . In the standard physicists’ notation, the respective equations are

$$(1.1) \quad \{\Omega, \Omega\} = 0 \quad \text{and} \quad (S, S) = 0,$$

where  $\Omega$  (by a widespread abuse of terminology) is the “BRST generator” in the Hamiltonian quantization and  $S$  is the master action in the Lagrangian quantization.

Under appropriate regularity conditions, the zero locus  $\mathcal{Z}_Q \subset \mathcal{N}$  of  $Q = \{\Omega, \cdot\}$  (of  $Q = (S, \cdot)$ ) is an odd Poisson manifold (respectively, a Poisson manifold) [4, 5], whose geometry captures crucial information about the theory on  $\mathcal{N}$ . In this paper, we mainly concentrate on *even* QP manifolds (which correspond to the BFV quantization and were implicit in [6]) because they have not been considered before; however, we formulate the general facts about the zero-locus reduction such that they apply to both even and odd QP manifolds. On an even QP manifold,  $\mathcal{Z}_Q$  carries an *anti*bracket; we then show that the equivalence classes of observables (the cohomology of  $Q$ ) are in a 1 : 1 correspondence with characteristic (Casimir) functions of the antibracket on  $\mathcal{Z}_Q$ , and gauge symmetries in the BFV theory on  $\mathcal{N}$  are Hamiltonian vector fields on  $\mathcal{Z}_Q$ .

Moreover, the zero locus  $\mathcal{Z}_{\mathbf{Q}}$  of the BFV differential on the extended phase space is a proper counterpart of the constraint surface in the following sense. In geometric terms, a first-class constrained system can be specified by its phase space (a symplectic manifold  $\mathcal{N}_0$ ) and the constraint surface  $\Sigma$ . On the extended phase space  $\mathcal{N}$  constructed in the BFV quantization, we can consider the dynamical system whose constraint surface, by definition, is  $\mathcal{Z}_{\mathbf{Q}}$  (in local coordinates on  $\mathcal{N}$ , the constraints can be chosen as the components of  $\mathbf{Q}$ ). Then the constrained systems  $(\mathcal{N}_0, \Sigma)$  and  $(\mathcal{N}, \mathcal{Z}_{\mathbf{Q}})$  are *equivalent*: the respective algebras of the equivalence classes of observables are naturally isomorphic as Poisson algebras.

Beyond the BRST context, algebras of functions on QP manifolds, which are differential Poisson algebras (associative supercommutative algebras endowed with a bracket operation and a differential that is a derivation of the bracket) can arise from complexes endowed with a super-commutative associative multiplication and a Gerstenhaber-like multiplication (“*bracket*”); the differential is then interpreted as the  $\mathbf{Q}$ -structure, and the bracket becomes the P-structure (the Poisson or the BV bracket on the dual (super)manifold). The basic examples are the cohomology complexes of a Lie algebra  $\mathfrak{a}$  with coefficients in  $\bigwedge \mathfrak{a}$  or  $\mathrm{Sa}$  (the exterior and symmetric tensor algebras); the general case involves  $L_\infty$  algebras [7].

In this algebraic context, reduction to the zero locus can yield relations between different complexes. In certain cases, the zero-locus reduction can be applied *repeatedly*; the equation ensuring the existence of a nilpotent vector field on the reduced manifold at the second step of the reduction can be the classical Yang–Baxter equation (CYBE), in which case the reduction leads to the well-known Sklyanin and Berezin–Kirillov brackets.

In addition to the usual QP manifolds, one can consider *bi-QP* manifolds, which are the geometric counterparts of bicomplexes, and in physical terms, originate in the BRST–anti-BRST ( $Sp(2)$ -symmetric/triplectic) quantization [8, 9, 10, 11]. With two BRST operators represented by two commuting (odd and nilpotent) vector fields, bi-QP manifolds might be called QQP manifolds; interestingly enough, the corresponding zero-locus reduction (to the submanifold on which both vector fields vanish) results in a “PP” manifold, i.e., gives rise to a *bi-Hamiltonian* structure. A typical example is obtained by starting with a Lie algebra  $\mathfrak{a}$  and deriving the second differential from a coalgebra structure. Compatibility between two differentials then implies that  $(\mathfrak{a}, \mathfrak{a}^*, \mathfrak{a} \oplus \mathfrak{a}^*)$  is a Manin triple [12]. There also exists an alternative construction of a bi-QP manifold from a *single* Lie algebra structure, which results in non-Abelian triplectic antibrackets [13] on the space of common zeroes of the differentials (and thus, the zero locus reduction leads to a nontrivial relation to the bicomplex used in the extended BRST symmetry).

This paper is organized as follows. In Sec. **2.2**, we recall the main points of the zero locus reduction on (odd or even) QP manifolds. Symmetries of QP manifolds are reviewed in Sec. **2.3**. In Sec. **3**, we turn to a more detailed analysis of even QP manifolds corresponding to the BFV

quantization. In Secs. **3.1**–**3.2**, we recall several facts about the BFV formalism in the form that is suitable for what follows. The results given in **3.4** state the relation between objects in the bulk of the phase space and on the zero locus submanifold. We briefly discuss in Sec. **3.5** how these results can be restated for the BV formalism. In Sec. **4**, we consider specific brackets resulting from the zero-locus reduction. In Sec. **5**, we study *bi*-QP manifolds.

## 2. GEOMETRY OF QP MANIFOLDS AND ZERO LOCUS REDUCTION

Geometric objects underlying the BRST quantization are the *QP* manifolds.

**2.1. Definition** ([3, 4]). *A QP manifold is a supermanifold  $\mathcal{N}$  equipped with a bracket  $\{\cdot, \cdot\}$  such that*

$$(2.1) \quad \begin{aligned} \{F, G\} &= -(-1)^{(\mathbf{p}(F)+\kappa)(\mathbf{p}(G)+\kappa)} \{G, F\}, \\ \{F, GH\} &= \{F, G\}H + (-1)^{(\mathbf{p}(F)+\kappa)\mathbf{p}(G)} G\{F, H\}, \\ \{F, \{G, H\}\} &= \{\{F, G\}, H\} + \{G, \{F, H\}\}(-1)^{(\mathbf{p}(F)+\kappa)(\mathbf{p}(G)+\kappa)}, \end{aligned}$$

for  $F, G, H \in \mathcal{F}_{\mathcal{N}}$  (smooth functions on  $\mathcal{N}$ ), and with an odd nilpotent vector field  $\mathbf{Q}$ ,  $\mathbf{Q}^2 = 0$ , such that

$$(2.2) \quad \mathbf{Q}\{F, G\} - \{\mathbf{Q}F, G\} - (-1)^{\mathbf{p}(F)+\kappa} \{F, \mathbf{Q}G\} = 0, \quad F, G \in \mathcal{F}_{\mathcal{N}}$$

(where  $\mathbf{p}(\cdot)$  is the Grassmann parity). *QP manifolds with a Poisson bracket ( $\kappa = 0$ ) are called even, and those with an antibracket ( $\kappa = 1$ ), odd.*

Odd *QP* manifolds arise in the BV quantization, and even ones in the BFV quantization. Odd *QP* manifolds were introduced in [3] and were studied in [4, 5]. In most of our definitions, *QP* manifolds can be either even or odd; in Sec. **3**, however, we concentrate on even *QP* manifolds, which have not been given enough attention previously.

**2.2. The zero locus of  $\mathbf{Q}$ .** In what follows,  $\mathcal{Z}_{\mathbf{Q}}$  denotes the zero locus of the odd vector field  $\mathbf{Q}$  on a *QP* manifold  $\mathcal{N}$ . We assume  $\mathcal{Z}_{\mathbf{Q}}$  to be a nonempty smooth submanifold and denote by  $\mathcal{I}_{\mathcal{Z}_{\mathbf{Q}}} \subset \mathcal{F}_{\mathcal{N}}$  the ideal of smooth functions vanishing on  $\mathcal{Z}_{\mathbf{Q}}$ .

The odd vector field  $\mathbf{Q}$  is called *regular* if each function  $f \in \mathcal{I}_{\mathcal{Z}_{\mathbf{Q}}}$  can be represented as

$$(2.3) \quad f = \sum_{\alpha} f_{\alpha} \mathbf{Q} \Gamma^{\alpha},$$

with some  $f_{\alpha}, \Gamma^{\alpha} \in \mathcal{F}_{\mathcal{N}}$  (i.e., if the components of  $\mathbf{Q}$  generate  $\mathcal{I}_{\mathcal{Z}_{\mathbf{Q}}}$ ). We say that a submanifold  $\mathcal{L} \subset \mathcal{N}$  is *coisotropic* if

$$(2.4) \quad \{\mathcal{I}_{\mathcal{L}}, \mathcal{I}_{\mathcal{L}}\} \subset \mathcal{I}_{\mathcal{L}}.$$

**2.2.1. Lemma.** *If  $\mathbf{Q}$  is regular,  $\mathcal{Z}_{\mathbf{Q}}$  is a coisotropic submanifold of the *QP* manifold  $\mathcal{N}$ .*

*Proof.* Let  $f, g \in \mathcal{F}_{\mathcal{N}}$  vanish on  $\mathcal{Z}_{\mathbf{Q}}$ . Using representation (2.3), the Leibnitz rule, Eq. (2.2), and nilpotency of  $\mathbf{Q}$ , we see that  $\{f_{\alpha}(\mathbf{Q}\Gamma^{\alpha}), (\mathbf{Q}\Gamma^{\beta})g_{\beta}\}|_{\mathcal{Z}_{\mathbf{Q}}} = (f_{\alpha}\{\mathbf{Q}\Gamma^{\alpha}, \mathbf{Q}\Gamma^{\beta}\}g_{\beta})|_{\mathcal{Z}_{\mathbf{Q}}} = 0$ .  $\square$

In what follows, we assume  $\mathcal{Z}_{\mathbf{Q}}$  to be coisotropic even in those cases where  $\mathbf{Q}$  is not regular.

The algebra  $\mathcal{F}_{\mathcal{Z}_{\mathbf{Q}}}$  of smooth functions on  $\mathcal{Z}_{\mathbf{Q}}$  is the quotient  $\mathcal{F}_{\mathcal{N}}/\mathcal{I}_{\mathcal{Z}_{\mathbf{Q}}}$ . We then have

**2.2.2. Lemma.** *There is a well-defined binary operation given by  $\{, \}_{\mathbf{Q}} : \mathcal{F}_{\mathcal{Z}_{\mathbf{Q}}} \times \mathcal{F}_{\mathcal{Z}_{\mathbf{Q}}} \rightarrow \mathcal{F}_{\mathcal{Z}_{\mathbf{Q}}}$*

$$(2.5) \quad \{f, g\}_{\mathbf{Q}} = \{F, \mathbf{Q}G\}|_{\mathcal{Z}_{\mathbf{Q}}}, \quad f, g \in \mathcal{F}_{\mathcal{Z}_{\mathbf{Q}}}, \quad F, G \in \mathcal{F}_{\mathcal{N}}, \quad F|_{\mathcal{Z}_{\mathbf{Q}}} = f, \quad G|_{\mathcal{Z}_{\mathbf{Q}}} = g,$$

where  $F$  and  $G \in \mathcal{F}_{\mathcal{N}}$  are viewed as representatives of functions on  $\mathcal{Z}_{\mathbf{Q}}$ . It makes  $\mathcal{Z}_{\mathbf{Q}}$  into a Poisson manifold.

The proof is a straightforward generalization of a proof given in [5]. It is obvious that the parity of the induced bracket on  $\mathcal{Z}_{\mathbf{Q}}$  is opposite to the parity of the  $\{, \}$  bracket on  $\mathcal{N}$ . An important characteristic of the differential  $\mathbf{Q}$  is the homology of the linear operators  $\mathbf{Q}_p : T_p\mathcal{N} \rightarrow T_p\mathcal{N}$ ,  $p \in \mathcal{Z}_{\mathbf{Q}}$ , defined as follows. We consider the tangent space  $T_p\mathcal{N}$  as the quotient of the vector fields  $\text{Vect}_{\mathcal{N}}$  modulo those that vanish at  $p$ . Then

$$(2.6) \quad \mathbf{Q}_p(x) = ([\mathbf{Q}, X])|_p, \quad X \in \text{Vect}_{\mathcal{N}}, \quad x = X_p \in T_p\mathcal{N}.$$

This operation is well-defined once  $\mathbf{Q}$  vanishes at  $p$ .

**2.2.3. Definition.** *A QP manifold  $\mathcal{N}$  is called proper if the homology of the linear operator  $\mathbf{Q}_p : T_p\mathcal{N} \rightarrow T_p\mathcal{N}$  is trivial at each point  $p \in \mathcal{Z}_{\mathbf{Q}}$ .*

This definition is equivalent to the one given in [4] (and [5]), but uses only invariant notions (in local coordinates  $\Gamma^A$ , we would have  $(\mathbf{Q}_p x)^A = (-1)^{p(x)+1} x^B \frac{\partial \mathbf{Q}^A}{\partial \Gamma^B}$ ). We now have

**2.2.4. Proposition** ([4, 5]). *Let  $\mathcal{N}$  be a proper QP manifold with a nondegenerate bracket. Then  $\mathcal{Z}_{\mathbf{Q}}$  is (anti)symplectic with respect to the induced bracket (2.5).*

One can replace  $\mathcal{Z}_{\mathbf{Q}}$  with a submanifold that still is coisotropic. As a straightforward generalization of 2.2.2, we have

**2.2.5. Theorem.** *Let  $\mathcal{N}$  be a QP manifold and  $\mathcal{L} \subset \mathcal{Z}_{\mathbf{Q}} \subset \mathcal{N}$  a coisotropic submanifold of  $\mathcal{N}$ . Then  $\mathcal{L}$  is a Poisson manifold<sup>1</sup> with the Poisson structure given by*

$$(2.7) \quad \{f, g\}_{\mathbf{Q}} = \{F, \mathbf{Q}G\}|_{\mathcal{L}}, \quad f, g \in \mathcal{F}_{\mathcal{L}}, \quad F, G \in \mathcal{F}_{\mathcal{N}}, \quad F|_{\mathcal{L}} = f, \quad G|_{\mathcal{L}} = g.$$

*Proof.* It is easy to see that (2.7) does not depend on the choice of representatives  $F, G \in \mathcal{F}_{\mathcal{N}}$  of  $f, g \in \mathcal{F}_{\mathcal{L}}$ . The Jacobi identity and the Leibnitz rule follow in the same way as for the bracket in Eq. (2.5), see [5].  $\square$

<sup>1</sup>By Poisson manifolds, we mean those with either an even bracket or an antibracket.

**2.3. Symmetries of QP manifolds** [5]. We now recall several basic facts about symmetries of QP structures on a manifold.

**2.3.1. Definition.** A vector field  $X$  on a QP manifold  $\mathcal{N}$  is called a symmetry of  $\mathcal{N}$  if it commutes with  $\mathbf{Q}$  and is a Poisson vector field, i.e.,

$$(2.8) \quad X\{F, G\} - \{XF, G\} - (-1)^{(\mathbf{p}(F)+\kappa)\mathbf{p}(X)}\{F, XG\} = 0, \quad F, G \in \mathcal{F}_{\mathcal{N}}.$$

Symmetries of the form  $X = \{\mathbf{Q}F, \cdot\}$  (with  $F \in \mathcal{F}_{\mathcal{N}}$ ) are called trivial.

The Lie algebras of symmetries and trivial symmetries behave in a very regular manner under the restriction to  $\mathcal{Z}_{\mathbf{Q}}$ .

**2.3.2. Proposition.** Let  $X$  be a symmetry of  $\mathcal{N}$ . Then  $X$  restricts to  $\mathcal{Z}_{\mathbf{Q}}$  and its restriction  $x$  is a Poisson vector field on  $\mathcal{Z}_{\mathbf{Q}}$  with respect to the bracket (2.7) on  $\mathcal{Z}_{\mathbf{Q}}$ , namely

$$(2.9) \quad x\{F, G\}_{\mathbf{Q}} - \{xF, G\}_{\mathbf{Q}} - (-1)^{(\mathbf{p}(F)+\kappa+1)\mathbf{p}(X)}\{F, xG\}_{\mathbf{Q}} = 0, \quad F, G \in \mathcal{F}_{\mathcal{Z}_{\mathbf{Q}}}.$$

If in addition  $X = \{\mathbf{Q}H, \cdot\}$  is a trivial symmetry,  $x$  is a Hamiltonian vector field with respect to the  $\{\cdot, \cdot\}_{\mathbf{Q}}$  bracket.

*Proof.* Any symmetry  $X$  restricts to  $\mathcal{Z}_{\mathbf{Q}}$  because  $XF|_{\mathcal{Z}_{\mathbf{Q}}} = 0$  for any  $F$  vanishing on  $\mathcal{Z}_{\mathbf{Q}}$ . Indeed, every such function can be represented as  $F = F_{\alpha} \cdot \mathbf{Q}\Gamma^{\alpha}$  with some functions  $F_{\alpha}$  and  $\Gamma^{\alpha}$ , provided  $\mathbf{Q}$  is regular. Because  $[X, \mathbf{Q}] = 0$ , we have  $XF|_{\mathcal{Z}_{\mathbf{Q}}} = ((XF_{\alpha})(\mathbf{Q}\Gamma^{\alpha}))|_{\mathcal{Z}_{\mathbf{Q}}} + (-1)^{\mathbf{p}(X)(\mathbf{p}(F_{\alpha})+1)}F_{\alpha}(\mathbf{Q}X\Gamma^{\alpha})|_{\mathcal{Z}_{\mathbf{Q}}} = 0$ . Equation (2.9) immediately follows from the definition of the zero locus bracket and the definition of symmetries. If in addition  $X = \{\mathbf{Q}H, \cdot\}$  is a trivial symmetry, for any function  $f \in \mathcal{F}_{\mathcal{Z}_{\mathbf{Q}}}$  we have

$$(2.10) \quad xf = X|_{\mathcal{Z}_{\mathbf{Q}}}f = \{\mathbf{Q}H, F\}|_{\mathcal{Z}_{\mathbf{Q}}} = (-1)^{\mathbf{p}(H)+\kappa+1}\{H|_{\mathcal{Z}_{\mathbf{Q}}}, f\}_{\mathbf{Q}},$$

where  $F \in \mathcal{F}_{\mathcal{N}}$  is a lift of  $f$  (i.e.,  $f = F|_{\mathcal{Z}_{\mathbf{Q}}}$ ) and  $\kappa$  is the parity of the  $\{\cdot, \cdot\}$  bracket. Thus,  $x = X|_{\mathcal{Z}_{\mathbf{Q}}}$  is a Hamiltonian vector field with respect to the bracket  $\{\cdot, \cdot\}_{\mathbf{Q}}$ .  $\square$

### 3. OBSERVABLES, GAUGE SYMMETRIES, AND ZERO LOCUS REDUCTION IN BFV AND BV QUANTIZATIONS

We now consider the embedding of a constrained system into the BFV extended theory with the BRST charge  $\Omega$  and study the “on-shell” gauge symmetries in the two descriptions of the same theory. In the Dirac (“non-extended”) formalism, the on-shell gauge symmetries are those nonvanishing on the constraint surface, and in the BFV extended formalism, these are symmetries nonvanishing on the zero locus  $\mathcal{Z}_{\mathbf{Q}}$ . We show that the former are mapped into the latter such that the equivalence classes of observables in the original theory are mapped into equivalence classes of observables in the BFV theory (the latter can be considered as gauge invariant functions on  $\mathcal{Z}_{\mathbf{Q}}$ ). In this sense, *the zero locus  $\mathcal{Z}_{\mathbf{Q}}$  plays the role of a constraint surface in the BFV theory.* We concentrate

on the BFV case, where we assume the phase space to be finite-dimensional; reformulation of our results for the BV quantization, although straightforward at the formal level, requires some care because the BV configuration space of any realistic model is infinite-dimensional (see **3.5**).

**3.1. A reminder on constrained dynamics.** We begin with recalling several basic facts about constrained dynamics in the form that will be suitable in what follows.

**3.1.1. Basics of the Dirac constrained dynamics.** We consider a first-class constrained Hamiltonian system, defined on a phase space (symplectic manifold)  $\mathcal{N}_0$  with the *constraints*  $T_\alpha$  (functions on  $\mathcal{N}_0$ ) such that

$$(3.1) \quad \{T_\alpha, T_\beta\} = U_{\alpha\beta}^\gamma T_\gamma,$$

where  $\{, \}$  is the Poisson bracket on  $\mathcal{N}_0$ . For simplicity, we assume the first-class constraints  $T_\alpha$  to be irreducible. Let  $\Sigma$  denote the *constraint surface*  $T_\alpha = 0$ . A geometrically invariant way to specify a first-class constrained system is to fix the pair  $(\mathcal{N}_0, \Sigma)$  (a symplectic manifold and a coisotropic submanifold). Different choices for  $T_\alpha$  then give different generators of the ideal of functions vanishing on  $\Sigma$ .

By definition, an *observable* is a function on  $\mathcal{N}_0$  satisfying  $\{A, T_\alpha\}|_\Sigma = 0$ . Under the standard regularity conditions, each function vanishing on  $\Sigma$  is proportional to the constraints, and therefore,

$$(3.2) \quad \{A, T_\alpha\} = A_\alpha^\beta T_\beta$$

for some functions  $A_\alpha^\beta$ . Observables vanishing on  $\Sigma$  are called *trivial*. Two observables are called *equivalent* if they differ by a trivial observable. The space of equivalence classes of observables is a Poisson algebra, i.e., is closed under multiplication and under the Poisson bracket (these operations are well-defined on the equivalence classes via representatives). This algebra can be conveniently thought of as a subalgebra in the algebra of functions on  $\Sigma$ .

*Infinitesimal gauge transformations*, or *gauge symmetries*, are the Hamiltonian vector fields  $X_0 = \{\phi_0, \cdot\}$ , where  $\phi_0 = \phi_0^\alpha T_\alpha$  is a trivial observable. Gauge symmetries form a Lie algebra with respect to the commutator. For any observable  $A$  and a gauge symmetry  $X_0 = \{\phi_0, \cdot\}$ , we have

$$(3.3) \quad X_0 A = \{\phi_0, A\} = \{\phi_0^\alpha T_\alpha, A\} = \phi_0^\alpha \{T_\alpha, A\} + T_\alpha \{\phi_0^\alpha, A\},$$

which vanishes on  $\Sigma$  because  $A$  is an observable. Therefore, gauge symmetries preserve equivalence classes of observables.

By the *on-shell gauge symmetries*, we mean the equivalence classes of gauge symmetries modulo those vanishing on the constraint surface  $\Sigma$ . On-shell gauge symmetries can also be viewed as a subalgebra in the algebra of vector fields on  $\Sigma$ . Equivalence classes of observables (viewed as functions on  $\Sigma$ ) are then represented by functions annihilated by on-shell gauge symmetries.

**3.1.2. Basics of the BFV/BRST approach.** In the BFV quantization, the extended phase space  $\mathcal{N}$  is an even QP manifold whose  $\mathbf{Q}$ -structure is given by  $\mathbf{Q} = \{\Omega, \cdot\}$ , where  $\Omega$  is a function on  $\mathcal{N}$  (called the BRST charge) satisfying  $\{\Omega, \Omega\} = 0$ . In applications, the BFV extended phase space is usually equipped with an additional structure, the ghost charge  $G \in \mathcal{F}_{\mathcal{N}}$ . Functions with a definite ghost number are eigenfunctions of the ghost number operator

$$(3.4) \quad g = \{G, \cdot\},$$

corresponding to integer eigenvalues. The BRST charge is required to have the ghost number 1,

$$(3.5) \quad \{G, \Omega\} = \Omega.$$

We now consider a QP manifold  $\mathcal{N}$  that is not necessarily constructed via the BFV prescription; however, we refer to the objects on  $\mathcal{N}$  as BFV ones because the applications in what follows will be to the case where  $\mathcal{N}$  does result from the BFV construction. This also helps to distinguish between observables and symmetries on the QP manifold and those in the initial theory (Sec. 3.1.1), with ‘BFV’ used to refer to the former.

A *BFV observable*  $A$  is a function on the QP manifold  $\mathcal{N}$  satisfying

$$(3.6) \quad \mathbf{Q}A = \{\Omega, A\} = 0, \quad \text{gh}(A) = 0.$$

The  $\mathbf{Q}$ -exact BFV observables are called *trivial*. Two BFV observables  $A$  and  $\tilde{A}$  are equivalent if  $A - \tilde{A} = \mathbf{Q}B$  for some function  $B$ ; the equivalence classes of observables are then the cohomology of  $\mathbf{Q}$  in the ghost number zero. The algebra of BFV observables is a Poisson algebra (multiplication and the Poisson bracket can be defined via representatives).

A vector field  $X$  is called a *BFV gauge symmetry* if  $X = \{\mathbf{Q}H, \cdot\}$  for some function  $H$  with  $\text{gh}(X) = \text{gh}(\mathbf{Q}H) = 0$  (these are trivial symmetries (see 2.3.1) of the corresponding QP manifold). In other words, BFV gauge symmetries are the Hamiltonian vector fields generated by trivial BFV observables. If  $A$  is an observable and  $X = \{\mathbf{Q}H, \cdot\}$  a BFV gauge symmetry, we see that  $XA = \{\mathbf{Q}H, A\} = \mathbf{Q}\{H, A\}$  is a trivial observable, i.e., BFV gauge symmetries preserve the equivalence classes of BFV observables.

**3.2. From Dirac to the BFV formulation of a constrained system.** Formal similarities between the Dirac and BFV formalisms are summarized in Table 1. We now make contact between 3.1.1 and 3.1.2 by taking the extended phase space  $\mathcal{N}$  and the BRST charge  $\Omega$  to be those arising in the BFV formalism from a given first-class constrained system  $(\mathcal{N}_0, \Sigma)$ . As before, the constraints  $T_\alpha \in \mathcal{F}_{\mathcal{N}_0}$  are taken to be irreducible; to construct the BFV formalism, one then introduces ghosts  $c^\alpha$ , with  $\text{gh}(c^\alpha) = 1$ ,  $\mathbf{p}(c^\alpha) = \mathbf{p}(T_\alpha) + 1$  and their conjugate momenta  $\mathcal{P}_\alpha$ ,

$$(3.7) \quad \{c^\alpha, \mathcal{P}_\beta\} = \delta^\alpha_\beta,$$

	Dirac (Sec. <b>3.1.1</b> )	BFV (Sec. <b>3.1.2</b> )
observables	$A_0, \{A_0, T_\alpha\} = A_\alpha^\beta T_\beta$	$\mathbf{Q}A = \{\Omega, A\} = 0, \text{gh}(A) = 0$
trivial observables	$(A_0) _\Sigma = 0$	$A = \mathbf{Q}B$
equivalent observables	$A_0 \sim A_0 + a^\alpha T_\alpha$	$A \sim A + \mathbf{Q}B$
gauge symmetries	$X_0 = \{\phi_0^\alpha T_\alpha, \cdot\}$	$X = \{\mathbf{Q}H, \cdot\}, \text{gh}(X) = 0$

TABLE 1.

with  $\text{gh}(\mathcal{P}_\alpha) = -1$ ,  $\mathbf{p}(\mathcal{P}_\alpha) = \mathbf{p}(T_\alpha) + 1$ . The extended phase space  $\mathcal{N}$  is the direct product of  $\mathcal{N}_0$  with the superspace spanned by  $c^\alpha$  and  $\mathcal{P}_\alpha$ .<sup>2</sup> The Poisson bracket on  $\mathcal{N}$  is the product Poisson bracket of that on  $\mathcal{N}_0$  and (3.7).

One introduces the ghost charge (where we assume the constraints to be bosonic to avoid extra sign factors)

$$(3.8) \quad G = c^\alpha \mathcal{P}_\alpha, \quad \{G, c^\alpha\} = c^\alpha, \quad \{G, \mathcal{P}_\alpha\} = -\mathcal{P}_\alpha.$$

The BRST charge  $\Omega$  is an odd function defined by the condition that it has the ghost number 1 and satisfies

$$(3.9) \quad \{\Omega, \Omega\} = 0$$

with the boundary condition

$$(3.10) \quad \Omega = c^\alpha T_\alpha + \dots,$$

where  $\dots$  means higher-order terms in the ghost momenta. It is well known [1, 14, 8, 16] that under standard assumptions, the BRST charge  $\Omega$  exists for any constrained system. Up to the first order in  $\mathcal{P}_\alpha$ , one has

$$(3.11) \quad \Omega = c^\alpha T_\alpha - \frac{1}{2} \mathcal{P}_\gamma U_{\alpha\beta}^\gamma c^\alpha c^\beta + \dots,$$

where the structure functions are those from (3.1).

As regards observables, the following statement is well known [1, 14] (see also [16]).

**3.2.1. Proposition.** *The algebra of the equivalence classes of observables on  $\mathcal{N}_0$  and the algebra of the equivalence classes of BFV observables (the cohomology of  $\mathbf{Q}$  in the ghost number zero) on the extended phase space  $\mathcal{N}$  are isomorphic as Poisson algebras.*

This means that if  $A_0 \in \mathcal{F}_{\mathcal{N}_0}$  is an observable of the constrained system on  $\mathcal{N}_0$ , there exists a BFV observable  $A \in \mathcal{F}_{\mathcal{N}}$  with  $\text{gh}(A) = 0$  such that

$$(3.12) \quad A|_{\mathcal{N}_0} = A_0.$$

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<sup>2</sup>When the constraints are defined *locally*, the extended phase space is a vector bundle over the original phase space, as, for example, in [15].



Moreover, two BFV observables corresponding to the same observable  $A_0$  differ by a trivial BFV observable. If in addition  $A_0$  is a trivial observable, it follows that  $A = \{\Omega, B\}$ . The Poisson bracket on  $\mathcal{N}$  induces a bracket on the cohomology of  $\mathbf{Q}$ , and one has

$$(3.13) \quad \{A, B\}|_{\mathcal{N}_0} = \{A_0, B_0\}.$$

The isomorphism between the BRST cohomology in the ghost number zero and the algebra of equivalence classes of observables of the constrained system on  $\mathcal{N}_0$  is given by the restriction of representatives to the initial constrained surface  $\Sigma \subset \mathcal{N}_0 \subset \mathcal{N}$  (recall that equivalence classes of observables are gauge invariant functions on  $\Sigma$ ).

It also follows from **3.2.1** that because gauge symmetries of the initial system  $(\mathcal{N}_0, \Sigma)$  are generated by trivial observables, each gauge symmetry can be lifted to a BFV gauge symmetry.

**3.3. Zero locus  $\mathcal{Z}_{\mathbf{Q}}$  in the BFV theory, the general case.** We now consider an even QP-manifold  $\mathcal{N}$  that is not necessarily constructed by the BFV procedure for a constrained system. We assume  $\mathcal{N}$  to be symplectic, and the odd nilpotent vector field  $\mathbf{Q}$  to be regular in the sense of **2.3**. The zero locus  $\mathcal{Z}_{\mathbf{Q}}$  is thus a coisotropic submanifold of  $\mathcal{N}$ . Because each trivial BFV observable  $A = \mathbf{Q}B$  vanishes on  $\mathcal{Z}_{\mathbf{Q}}$ , each cohomology class uniquely determines a function on  $\mathcal{Z}_{\mathbf{Q}}$ . Thus, there is a mapping

$$(3.14) \quad H_{\mathbf{Q}}^0 \rightarrow \mathcal{F}_{\mathcal{Z}_{\mathbf{Q}}}$$

from the space of inequivalent observables to functions on  $\mathcal{Z}_{\mathbf{Q}}$ .

In what follows, we say that a statement holds locally if it is true in every sufficiently small neighbourhood. Mapping (3.14) is locally an embedding in view of the following proposition.

**3.3.1. Proposition.** *Let  $\mathbf{Q} = \{\Omega, \cdot\}$  be regular in the sense of **2.2**. Locally, each BFV observable vanishing on  $\mathcal{Z}_{\mathbf{Q}}$  is a trivial BFV observable.*

*Proof.* Let  $A$  be an observable vanishing on  $\mathcal{Z}_{\mathbf{Q}}$ , i.e.,  $\mathbf{Q}A = 0$ ,  $A|_{\mathcal{Z}_{\mathbf{Q}}} = 0$ . We must show that  $A = \mathbf{Q}X$  in a sufficiently small neighbourhood  $U$  of any point  $p \in \mathcal{Z}_{\mathbf{Q}}$ . It is well known that locally there exists a coordinate system  $p_i, q^j, p_\alpha, q^\beta, c^\alpha, \mathcal{P}_\beta$  on  $\mathcal{N}$  such that

$$(3.15) \quad \begin{aligned} \Omega &= p_i c^i, \\ \{q^i, p_j\} &= \delta_j^i, \quad \{q^\alpha, p_\beta\} = \delta_\beta^\alpha, \quad \{c^\alpha, \mathcal{P}_\beta\} = \delta_\beta^\alpha. \end{aligned}$$

Since the function  $A$  vanishes on  $\mathcal{Z}_{\mathbf{Q}}$ , it can be represented as

$$(3.16) \quad A = A^\alpha p_\alpha + A_\alpha c^\alpha.$$

Now the odd vector field  $\mathbf{Q}$  becomes

$$(3.17) \quad \mathbf{Q} = -c^\alpha \frac{\partial}{\partial q^\alpha} + p_\alpha \frac{\partial}{\partial \mathcal{P}_\alpha},$$

and can be considered as the exterior differential under the identification  $c^\alpha = -dq^\alpha$ ,  $p_\alpha = d\mathcal{P}_\alpha$ , while  $A$  becomes a 1-form. The assertion immediately follows from the super analogue of the Poincaré lemma in  $U$ .  $\square$

**3.3.2. Embedding  $H_Q^0$  into functions on  $\mathcal{Z}_Q$ .** As before,  $\mathcal{Z}_Q$  is the zero locus submanifold of  $Q = \{\Omega, \cdot\}$ . We recall from 2.3.2 that each BfV gauge symmetry  $X$  can be restricted to  $\mathcal{Z}_Q$  and the restriction  $x = X|_{\mathcal{Z}_Q}$  is a  $\{\cdot, \cdot\}_Q$ -Hamiltonian vector field on  $\mathcal{Z}_Q$ . The image of BfV gauge symmetries under the restriction to  $\mathcal{Z}_Q$  is called the algebra of the *on-shell BfV symmetries*. The functions on  $\mathcal{Z}_Q$  that are annihilated by the on-shell BfV symmetries are then the characteristic functions of the  $\{\cdot, \cdot\}_Q$  antibracket on  $\mathcal{Z}_Q$ .<sup>3</sup>

Because BfV observables are annihilated by BfV gauge symmetries, the restriction to  $\mathcal{Z}_Q$  maps BfV observables into characteristic functions of  $\{\cdot, \cdot\}_Q$ . For the equivalence classes of BfV observables (the cohomology of  $Q$ ), this mapping is certainly an embedding locally. It is also an isomorphism in the important case of a BfV QP manifold considered in 3.4. Locally, we choose flat coordinates in some neighborhood  $U$  of a point  $p \in \mathcal{Z}_Q$  and use explicit form (3.15) of the BRST charge  $\Omega$  and the Poisson bracket to arrive at

**3.3.3. Theorem.** *Locally, the equivalence classes of BfV observables (the cohomology of  $Q$ ) are in a 1 : 1 correspondence with characteristic functions of the  $\{\cdot, \cdot\}_Q$  antibracket on  $\mathcal{Z}_Q$ .*

We note that in one direction, this statement holds in general (i.e., not only locally) because for any observable  $A$ , we have

$$(3.18) \quad \{f, A|_{\mathcal{Z}_Q}\}_Q = \{F, QA\}|_{\mathcal{Z}_Q} = 0,$$

where  $F \in \mathcal{F}_\mathcal{N}$  is the lift of a function  $f \in \mathcal{F}_{\mathcal{Z}_Q}$  and  $A|_{\mathcal{Z}_Q}$  is the image of  $A$  under (3.14). Thus,  $A|_{\mathcal{Z}_Q}$  is a characteristic function of the antibracket  $\{\cdot, \cdot\}_Q$  on  $\mathcal{Z}_Q$ .

The “ $\mathcal{Z}_Q$ -based” view on the BfV formalism developed here can be expressed as follows. *Any even QP manifold  $\mathcal{N}$  gives rise to the constrained system  $(\mathcal{N}, \mathcal{Z}_Q)$ , i.e., a constrained system whose phase space is  $\mathcal{N}$  and the constrained surface is  $\mathcal{Z}_Q$ .* We recall from 3.1.1 that gauge transformations and the algebra of observables can be reconstructed if a first-class constrained system is specified in geometric terms, via its phase space (a symplectic manifold) and the constraint surface (a coisotropic submanifold). We now take this pair to be  $(\mathcal{N}, \mathcal{Z}_Q)$  (with  $\mathcal{Z}_Q$  being coisotropic in view of 2.2.1). In local coordinates, the constraints are the components of  $Q$ ; in a neighborhood  $U \subset \mathcal{N}$ , the following statement is obvious in the special coordinates in which  $\Omega$  and  $\{\cdot, \cdot\}_Q$  are given by (3.15).

**3.3.4. Theorem.** *On a QP manifold  $\mathcal{N}$ , the constrained system  $(\mathcal{N}, \mathcal{Z}_Q)$  is locally equivalent to the BfV theory on the extended phase space  $\mathcal{N}$  with the BRST charge  $\Omega$  (i.e., the respective algebras of equivalence classes of observables are isomorphic as Poisson algebras).*

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<sup>3</sup>We recall that a function  $f$  on the (odd) Poisson manifold  $\mathcal{N}$  is said to be a characteristic (Casimir) function of an (odd) Poisson bracket  $\{\cdot, \cdot\}$  if  $\{f, H\} = 0$  for any function  $H$ .

In a more physical language, the equivalence can be reformulated by saying that the two constrained dynamics are equivalent.

The above considerations show that BFV observables are related to  $\mathcal{Z}_Q$  in the same way as observables in the initial theory (Sec. 3.1.1) are related to the constraint surface  $\Sigma$ . This allows us to interpret  $\mathcal{Z}_Q$  as the extended constraint surface. In the general case, this correspondence takes place at the local level only.

**3.4. Zero locus  $\mathcal{Z}_Q$  in the BFV formulation of a constrained system.** We now concentrate on the important case where the QP manifold under consideration is a BFV extended phase space obtained by the BFV procedure from a given constrained system  $(\mathcal{N}_0, \Sigma)$ .

**3.4.1. Proposition.** *The initial constraint surface  $\Sigma \subset \mathcal{N}_0$  is a submanifold of the zero locus  $\mathcal{Z}_Q \subset \mathcal{N}$  of the BRST differential  $Q = \{\Omega, \cdot\}$ .*

*Proof.* We restrict ourselves to an irreducible theory with constraints  $T_\alpha$  (although the statement also is true for reducible constraints); the structure of the BRST charge is then given by (3.10). Considered as a submanifold in  $\mathcal{N}$ , the initial phase space  $\mathcal{N}_0$  is determined by the equations  $c^\alpha = 0$  and  $\mathcal{P}_\alpha = 0$ . It follows from (3.10) and from  $\text{gh}(\Omega) = 1$  that the zero locus  $\mathcal{Z}_Q$  is determined by the equations

$$(3.19) \quad T_\alpha + \dots = 0, \quad \dots = 0,$$

where  $\dots$  denotes terms vanishing on  $\mathcal{N}_0$ . Then the intersection  $\mathcal{Z}_Q \cap \mathcal{N}_0$  (considered as a submanifold in  $\mathcal{N}_0$ ) is determined by the equations  $T_\alpha = 0$ , and therefore, coincides with the initial constraint surface  $\Sigma$ . Thus,  $\Sigma$  is a submanifold in  $\mathcal{Z}_Q$ .  $\square$

The zero locus can be described somewhat more explicitly if we recall that in the BFV formalism, functions on the extended phase space are formal power series in the ghost variables  $c^\alpha$  and  $\mathcal{P}_\alpha$ . This means that  $\mathcal{Z}_Q$  is actually determined by the equations

$$(3.20) \quad T_\alpha = 0, \quad c^\alpha = 0.$$

This, in its turn, gives an explicit construction of the antibracket  $\{\cdot, \cdot\}_Q$  on  $\mathcal{Z}_Q$ . Let  $y^i$  be local coordinates on  $\Sigma$ . Then  $y^i$  and  $\mathcal{P}_\alpha$  can be considered as local coordinates on  $\mathcal{Z}_Q$ . Evaluating (2.5), we now obtain

$$(3.21) \quad \{y^i, y^j\}_Q = 0, \quad \{\mathcal{P}_\alpha, y^i\}_Q = R_\alpha^i(y), \quad \{\mathcal{P}_\alpha, \mathcal{P}_\beta\}_Q = U_{\alpha\beta}^\gamma(y) \mathcal{P}_\gamma,$$

where  $R_\alpha^i(y) = \{T_\alpha, y^i\}|_\Sigma$  and  $U_{\alpha\beta}^\gamma(y) = U_{\alpha\beta}^\gamma|_\Sigma$  with  $U_{\alpha\beta}^\gamma$  from (3.11).

Using the explicit form (3.21) of the antibracket on  $\mathcal{Z}_Q$ , it is easy to describe its characteristic functions in terms of the initial constraint surface  $\Sigma$ . The following statement is obvious for irreducible constraints  $T_\alpha$  and can be easily generalized to reducible constraints.

**3.4.2. Proposition.** *Characteristic functions of the antibracket  $\{, \}_Q$  are in a 1 : 1 correspondence with gauge invariant functions on  $\Sigma$ .*

On a QP manifold constructed in accordance with the BFV prescription, the relation between the BRST cohomology and the geometry of the extended constrained surface  $\mathcal{Z}_Q$  can be made more precise than in the previous section. In particular, the respective counterparts of statements **3.3.1**, **3.3.3**, and **3.3.4** hold globally. We first see that (3.14) is an embedding.

**3.4.3. Proposition.** *On a QP manifold  $\mathcal{N}$  constructed in the BFV formalism, each BFV observable that vanishes on  $\mathcal{Z}_Q \subset \mathcal{N}$  is a trivial BFV observable.*

*Proof.* Let  $A$  be a BFV observable and  $A|_{\mathcal{Z}_Q} = 0$ . According to **3.4.1**,  $\Sigma \subset \mathcal{Z}_Q$ . Then  $A|_{\mathcal{Z}_Q} = 0$  implies  $A|_{\Sigma} = 0$  (a trivial observable). By **3.2.1**,  $A$  is a trivial BFV observable.  $\square$

We now consider the QP manifold constructed in the BFV formalism. Combining **3.4.3** with the argument given after **3.3.3** proves the next theorem in one direction; the other direction follows because each characteristic function on  $\mathcal{Z}_Q$  can be lifted to a BFV observable on  $\mathcal{N}$ , see **3.2.1** and **3.4.2**.

**3.4.4. Theorem.** *Equivalence classes of BFV observables (the cohomology of  $Q$  with the ghost number zero) on the BFV QP manifold are in a 1 : 1 correspondence with characteristic functions of the zero locus antibracket on  $\mathcal{Z}_Q$ .*

As in **3.3.3**, we now consider the extended phase space of the BFV formulation as the phase space of a “new” constrained system determined by the constraint surface  $\mathcal{Z}_Q$ . With  $\mathcal{N}$  in its turn obtained from a constrained dynamical system  $(\mathcal{N}_0, \Sigma)$  in accordance with the BFV formalism, we have a global version of **3.3.4**.

**3.4.5. Theorem.** *Let  $\mathcal{N}$  be a QP manifold constructed in the BFV formalism. The constrained system determined by the pair  $(\mathcal{N}, \mathcal{Z}_Q)$  is equivalent to the BFV theory on  $\mathcal{N}$  (i.e., the respective algebras of equivalence classes of observables are isomorphic as Poisson algebras).*

Combining this with **3.2.1**, we obtain a remarkable relation between the constrained systems specified by the respective pairs  $(\mathcal{N}_0, \Sigma)$  and  $(\mathcal{N}, \mathcal{Z}_Q)$ :

**3.4.6. Corollary.** *The constrained systems  $(\mathcal{N}_0, \Sigma)$  and  $(\mathcal{N}, \mathcal{Z}_Q)$  are equivalent (the respective algebras of inequivalent observables are isomorphic as Poisson algebras).*

We also note a difference between the initial and the extended constraint surfaces  $\Sigma$  and  $\mathcal{Z}_Q$  in that  $\Sigma$  carries an action of the gauge generators  $\{T_i, \cdot\}$ , while  $\mathcal{Z}_Q$  is equipped with the zero locus antibracket. This is not unnatural, because the on-shell gauge symmetries are Hamiltonian vector

fields with respect to the zero-locus antibracket, while inequivalent observables are (identified with) the characteristic functions of the zero-locus antibracket.<sup>4</sup>

Finally, we note that there is a slightly different point of view on the interpretation of BFV observables in terms of the geometry of  $\mathcal{Z}_Q$ . Namely, to each (odd) Poisson structure, one can associate the coboundary operator (differential) acting on antisymmetric tensor fields, with the action being the adjoint action of the Poisson bivector with respect to the Schouten-Nijenhuis bracket. Inequivalent observables are then the *zero-degree cohomology* of this differential on  $\mathcal{Z}_Q$  (tensors of zero degree are functions).

### 3.5. Observables, gauge symmetries, and zero locus reduction in the BV quantization.

The above can be reformulated for odd QP manifolds/BV quantization. In the BV formulation, the zero locus of  $Q = (S, \cdot)$ , where  $S$  is the master action, is the stationary surface of  $S$  (provided the BV antibracket  $(\cdot, \cdot)$  is nondegenerate). The BV observables are the cohomology of  $Q$  in the ghost number zero. The BV gauge symmetries are the vector fields of the form

$$(3.22) \quad X = (QB, \cdot),$$

and, thus, are Hamiltonian vector fields generated by trivial observables. Whenever the master action  $S$  is constructed via the BV prescription starting from a given initial action  $S_0$ , the zero locus of  $Q = (S, \cdot)$  is a certain extension of the stationary surface of the initial action  $S_0$ .

At the formal level, all the statements considered in the BFV scheme have their counterparts in the BV formalism. We do not restate here the contents of **3.1–3.4** for the odd case and refer instead to [5].<sup>5</sup> We only point out one important difference. Unlike the Hamiltonian picture, the Lagrangian one can be considered in the scope of a finite dimensional analogue only formally. The finite dimensional configuration space (the space of field histories) does not correspond to any physically relevant system. Thus all the BV counterparts of the statements of the previous section should be considered with some care. In particular, the BV quantization prescription requires the master action  $S$  to be a proper solution to the master equation. The condition imposed on the master action to be proper has no counterpart in the Hamiltonian picture. It implies that the corresponding configuration space is a proper QP manifold (which in general is not the case for the BFV phase space). In the finite dimensional case, this in turn implies that all the observables (the

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<sup>4</sup>This applies at the classical level. The notion of the initial and the extended constraint surfaces is essentially classical and has no obvious counterparts at the quantum level. At the quantum level, restrictions to the constraint surface should be understood as restriction to some quotient of the full Hilbert space of the quantum system. We do not discuss this very interesting subject here, and refer instead to [17], where a related problem was considered. We thank I.A. Batalin for an illuminating discussion of this point.

<sup>5</sup>To avoid misunderstanding, we note that we have changed our point of view on how the BFV/BV gauge symmetries should be *defined*: the BV gauge symmetries were called the “trivial gauge symmetries of the master action” in [5]; translating the results proved in [5] into the present conventions, therefore, requires some care with the “obsolete” definitions.

cohomology of  $\mathcal{Q}$ ) are trivial (except those of a topological nature). The  $\mathcal{Q}$  cohomology becomes nontrivial only when evaluated on space-time local functionals [18, 19].

#### 4. TOWERS OF BRACKETS

In this section, we study the possibility of a “second” zero-locus reduction, i.e., the reduction on a QP manifold which itself is the result of a zero-locus reduction. This leads to several well-known structures, including the classical Yang–Baxter equation.

**4.1. A “second” zero-locus reduction.** On a QP manifold  $\mathcal{N}$  (which can be either even or odd), a coisotropic submanifold  $\mathcal{L} \subset \mathcal{Z}_{\mathcal{Q}}$  (for example, a Lagrangian submanifold in  $\mathcal{N}$ ) is a P-manifold, i.e., is equipped with an (even or odd) Poisson structure (see 2.2.5). One can try to equip  $\mathcal{L}$  with a compatible  $\mathcal{Q}$  structure, thereby making it into a QP manifold. On a general QP manifold  $\mathcal{N}$ , there is no canonical structure inducing a  $\mathcal{Q}$  operator on  $\mathcal{L}$ . Instead, we can look for a  $\mathcal{Q}$  operator on  $\mathcal{L}$  in the form  $\mathcal{Q}_{\mathcal{L}} = \{H, \cdot\}_{\mathcal{Q}}$ , where  $\{, \}_{\mathcal{Q}}$  is the bracket given by (2.7) and  $H$  is a solution of the equation

$$(4.1) \quad \{H, H\}_{\mathcal{Q}} = 0, \quad H \in \mathcal{F}_{\mathcal{L}}, \quad \mathfrak{p}(H) = \mathfrak{p}(\{, \}_{\mathcal{Q}}) + 1.$$

Whenever such an  $H$  is found,  $\mathcal{L}$  becomes a QP manifold. With this  $\mathcal{Q}$ -structure, we can repeat the procedure, thereby producing a sequence of QP manifolds.

This construction can be restated in terms of differential Poisson algebras (the algebras of functions on QP manifolds). Even “more algebraically,” we consider the case where a differential Poisson algebra arises from a *complex* endowed with a super-commutative associative multiplication and a Gerstenhaber-like multiplication (see the Appendix). To these differential Poisson algebras, we can then apply one or more zero-locus reduction steps, resulting in relations between different complexes.

**4.2. Examples of the zero locus reduction on an even QP manifold.** Let  $\mathcal{M}$  be a cotangent bundle  $\mathcal{M} = T^*\mathcal{X}$ . We then write  $(q^a, p_a)$  for local coordinates on  $\mathcal{M}$  (which we take to be bosonic to avoid extra sign factors); the Poisson bracket then is  $\{q^a, p_b\} = \delta_b^a$ . We assume a Hamiltonian action of a Lie algebra  $\mathfrak{a}$  on  $\mathcal{M}$ . For simplicity, we consider the Hamiltonian action that is the lift of an action on  $\mathcal{X}$  via the vector fields  $X_i = X_i^a \frac{\partial}{\partial q^a}$ , with  $[X_i, X_j] = C_{ij}^k X_k$ . The generators of the Hamiltonian action on  $\mathcal{M}$  are then given by  $T_i = -p_a X_i^a(q)$ . Applying the BFV scheme to the constraints  $T_i$  gives the BRST generator

$$(4.2) \quad \Omega = -p_a X_i^a(q) \theta^i - \frac{1}{2} C_{ij}^k \xi_k \theta^i \theta^j.$$

We now take the submanifold  $\mathcal{L} \subset \mathcal{Z}_{\mathcal{Q}}$  (which is Lagrangian in  $\mathcal{M}$ ) determined by  $\theta^i = 0$  and  $p_a = 0$  and view  $q^a$  and  $\xi_i$  as local coordinates on  $\mathcal{L}$ . The antibracket  $(, ) \equiv \{, \}_{\mathcal{Q}}$  from 2.2.5 is then given by

$$(4.3) \quad (\xi_i, \xi_j) = C_{ij}^k \xi_k, \quad (q^a, \xi_i) = -X_i^a.$$

Using this antibracket structure on  $\mathcal{L}$ , we consider the equation

$$(4.4) \quad (H, H) = 0$$

for an even function  $H \in \mathcal{F}_{\mathcal{L}}$ . Given a solution  $H$ , we can construct the odd nilpotent vector field  $\mathbf{Q} = (H, \cdot)$  that makes  $\mathcal{L}$  into a QP manifold.

We consider solutions to (4.4) of the form

$$(4.5) \quad H_{\text{YB}} = -\frac{1}{2} r^{ij} \xi_i \xi_j,$$

where  $r$  is a skew-symmetric matrix with entries from  $\mathcal{F}_{\mathcal{X}}$ . Explicitly, Eq. (4.4) is the following generalization of the CYBE:

$$(4.6) \quad r^{\ell[i} C_{\ell m}^k r^{j]m} + X_{\ell}^a r^{\ell[i} \frac{\partial}{\partial q^a} r^{jk]} = 0.$$

We now proceed with the next stage of the zero locus reduction. The zero locus of the “Yang–Baxter differential”  $\mathbf{Q}_{\text{YB}} = (H_{\text{YB}}, \cdot)$  is determined by  $r^{ij} \xi_j = 0$ . We choose a smaller submanifold  $\mathcal{X} \subset \mathcal{Z}_{\mathbf{Q}_{\text{YB}}}$  determined by  $\xi_i = 0$ . Whenever (4.4) is satisfied,  $\{\cdot, \cdot\} = (\cdot, \mathbf{Q}_{\text{YB}} \cdot)$  is a Poisson bracket on  $\mathcal{X}$ . Explicitly, the Poisson brackets are given by

$$(4.7) \quad \{q^a, q^b\} = X_i^a r^{ij} X_j^b.$$

**4.2.1. The classical Yang–Baxter equation.** Antibracket (4.3) considered on  $q$ -independent functions coincides with the Schouten bracket on  $\bigwedge \mathfrak{a}$  viewed as the Grassmann algebra generated by  $\xi_i$ . In the case where  $r^{ij}$  is a constant matrix, (4.6) becomes the CYBE

$$(4.8) \quad r^{j[i} C_{jl}^k r^{m]l} = 0.$$

For each  $r^{ij}$  satisfying (4.8), the corresponding differential  $\mathbf{Q}_{\text{YB}}$  (considered on  $\bigwedge \mathfrak{a}$ ) is nothing but the cohomology differential of the Lie algebra complex with trivial coefficients (see Appendix A), for the Lie algebra defined on  $\mathfrak{a}^*$  by the structure constants  $F_k^{ij} = r^{il} C_{lk}^j - r^{jl} C_{lk}^i$ .

**4.2.2. The Sklyanin bracket.** With  $\mathcal{X}$  taken to be the Lie group corresponding to the Lie algebra  $\mathfrak{a}$ , we have two natural ways to define the action of  $\mathfrak{a}$  on  $\mathcal{X}$ , by the left- and right-invariant vector fields  $L_i$  and  $R_i$ . Proceeding along the steps described in the previous paragraphs with  $X_i^a$  taken to be  $L_i^a$  or  $R_i^a$ , we arrive at two Poisson brackets on  $\mathcal{X}$ ,

$$(4.9) \quad \{q^a, q^b\}_{\text{right}} = L_i^a r^{ij} L_j^b \quad \text{and} \quad \{q^a, q^b\}_{\text{left}} = R_i^a r^{ij} R_j^b,$$

which are compatible in view of  $[R_i, L_j] = 0$ . The Poisson bracket

$$(4.10) \quad \{q^a, q^b\}_{\text{Sklyanin}} = \{q^a, q^b\}_{\text{right}} - \{q^a, q^b\}_{\text{left}}$$

makes the Lie group  $\mathcal{X}$  into a Poisson–Lie group.

**4.3. Zero locus reduction on an odd QP manifold.** To reformulate the above for an odd QP manifold, we construct the BV scheme starting with a manifold  $\mathcal{X}$  with an  $\mathfrak{a}$  action. The  $\xi_i$  variables are then even, and because of the symmetry properties, the “tower of reductions” is shorter than for odd  $\xi_i$ . We then introduce antifields  $q_a^*$ , ghosts  $\theta^i$ , and their antifields  $\xi_i$ , with  $(\theta^i, \xi_j) = \delta_j^i$  (where restored the traditional notation for the antibracket). The differential

$$(4.11) \quad \mathbf{Q} = (S, \cdot), \quad S = q_a^* X_i^a \theta^i - \frac{1}{2} \xi_k C_{ij}^k \theta^i \theta^j$$

corresponds to the quantization of a theory with the vanishing classical action.

We choose a Lagrangian subspace  $\mathcal{L} \subset \mathcal{Z}_{\mathbf{Q}}$  determined by  $\theta^i = 0$  and  $q_a^* = 0$ . In accordance with Sec. 2, the zero locus reduction induces a Poisson bracket  $\{, \}_{\mathbf{Q}}$  on  $\mathcal{L}$  with the nonvanishing components

$$(4.12) \quad \{\xi_i, \xi_j\}_{\mathbf{Q}} = C_{ij}^k \xi_k, \quad \{q^a, \xi_i\}_{\mathbf{Q}} = X_i^a.$$

Unless  $\mathcal{X}$  is a *supermanifold*,  $\mathcal{L}$  is a purely even manifold, and therefore, the new generating equation with respect to the  $\{, \}_{\mathbf{Q}}$ -bracket has only the trivial solution. The tower of brackets is thus terminated.

We now recall that the *even* variables  $\xi_i$  generate the algebra of functions on  $\mathfrak{a}^*$ . Restricting ourselves to functions that are independent of the coordinates on  $\mathcal{X}$ , we see that (4.12) becomes the Berezin–Kirillov bracket on  $\mathfrak{a}^*$ ,

$$(4.13) \quad \{f, g\} = f \overleftarrow{\frac{\partial}{\partial \xi_i}} \xi_k C_{ij}^k \frac{\partial}{\partial \xi_j} g.$$

**4.3.1. Linear and nonlinear brackets.** The bracket in (4.13) is “linear” in the sense of its explicit dependence on  $\xi_i$ . For a Lie algebra  $\mathfrak{a}$ , one can construct “nonlinear” brackets  $\overleftarrow{\frac{\partial}{\partial \xi_i}} \Omega_{ij} \frac{\partial}{\partial \xi_j}$  on  $\mathfrak{a}^*$ , where the expansion of  $\Omega_{ij}$  in  $\xi_i$  starts with  $\xi_k C_{ij}^k$ . For a given bracket of this form, a natural problem is whether it can be transformed into the Berezin–Kirillov bracket by a change of coordinates. With the help of the zero-locus reduction, this is solved as follows. The Poisson bracket is represented as the zero-locus reduction of the *canonical* antibracket on a QP manifold with  $\mathbf{Q}$  determined by the Hamiltonian  $H = \Omega_{ij}(\xi) \theta^i \theta^j$ . The Jacobi identity for the Poisson bracket is rewritten as the master equation for  $H$ , and moreover, the terms containing higher powers of  $\xi_i$  are closed with respect to the differential  $\mathbf{Q}_0 = \{H_0, \cdot\}$ , where  $H_0 = \xi_k C_{ij}^k \theta^i \theta^j$  is the “linear” part of the Hamiltonian. We thus have proved the fact known from other considerations (and in a more powerful analytic version) [20]

**4.3.2. Corollary.** *Let  $\Omega_{ij}(\xi) = \xi_k C_{ij}^k + \xi_k \xi_l C_{ij}^{kl} + \dots$  be the matrix of a Poisson bracket on  $\mathfrak{a}^*$ , where  $C_{ij}^k$  are the structure constants of a Lie algebra  $\mathfrak{a}$ . Then  $\Omega_{ij}(\xi)$  can be reduced to the form  $\xi_k C_{ij}^k$  by a change of variables  $\xi_i \mapsto f_i(\xi)$  if the second cohomology group  $H^2(\mathfrak{a}, \mathbf{S}\mathfrak{a})$  of  $\mathfrak{a}$  with coefficients in  $\mathbf{S}\mathfrak{a}$  is trivial.*

Similar considerations in the BFM case lead to similar statements for the nonlinear antibracket.



## 5. BI-QP MANIFOLDS

Up to now, we have studied QP manifolds whose differential corresponds to a *single* solution of the corresponding “master” equation. We now consider bi-QP manifolds.

**5.1. A BFV-like formulation of the bialgebra complex.** In the previous section, we associated an even QP manifold with a vector space  $\mathfrak{a}$  and a smooth manifold  $\mathcal{M} = T^*\mathcal{X}$ . Namely, a Lie algebra structure on  $\mathfrak{a}$  and the vector fields  $X_i$  (giving an  $\mathfrak{a}$ -module structure on  $\mathcal{F}_{\mathcal{X}}$ ) can be read off from a solution of the generating equation

$$(5.1) \quad \{\Omega, \Omega\} = 0$$

with the ansatz (4.2). The algebra of functions on the thus constructed QP manifold is  $\mathfrak{A} = \text{Hom}(\bigwedge \mathfrak{a}, \bigwedge \mathfrak{a}) \otimes \mathcal{F}_{T^*\mathcal{X}}$ ; we interpret  $\text{Hom}(\bigwedge \mathfrak{a}, \bigwedge \mathfrak{a})$  as the algebra generated by the odd variables  $\theta^i$  and  $\xi_j$ . The basic Poisson bracket relations are

$$(5.2) \quad \{\theta^i, \xi_j\} = \delta_j^i, \quad \{q^a, p_b\} = \delta_b^a,$$

where  $q, p$  are the standard local coordinates on the cotangent bundle  $\mathcal{M} = T^*\mathcal{X}$ . We have the solution

$$(5.3) \quad C = -p_a X_i^a(q) \theta^i - \frac{1}{2} \xi_k C_{ij}^k \theta^i \theta^j.$$

At the same time, every solution of (5.1) of the form

$$(5.4) \quad F = -p_a X^{ia} \xi_i - \frac{1}{2} \theta^k F_k^{ij} \xi_i \xi_j$$

determines a coalgebra structure on the vector space  $\mathfrak{a}$ , or equivalently, a Lie algebra structure on  $\mathfrak{a}^*$ , and makes  $\mathcal{F}_{\mathcal{X}}$  into an  $\mathfrak{a}^*$ -module, with the vector fields  $X^i = R^{ia} \frac{\partial}{\partial q^a} \in \text{Vect}_{\mathcal{X}}$  representing the action of the basis elements of  $\mathfrak{a}^*$ . Then  $\mathfrak{A}$  is equipped with Poisson bracket (5.2) and the differentials

$$(5.5) \quad \begin{aligned} d_C &= \{C, \cdot\} \\ &= -\frac{1}{2} C_{ij}^k \theta^i \theta^j \frac{\partial}{\partial \theta^k} - \xi_k C_{ij}^k \theta^i \frac{\partial}{\partial \xi_j} - p_a X_i^a \frac{\partial}{\partial \xi_i} + \theta^i X_i^a \frac{\partial}{\partial q^a} - \theta^i p_a X_{i,b}^a \frac{\partial}{\partial p_b}, \\ d_F &= \{F, \cdot\} \\ &= -\frac{1}{2} F_k^{ij} \xi_i \xi_j \frac{\partial}{\partial \xi_k} - \theta^k F_k^{ij} \xi_i \frac{\partial}{\partial \theta^j} - p_a X^{ia} \frac{\partial}{\partial \theta^i} + \xi_i X^{ia} \frac{\partial}{\partial q^a} - \xi_i p_a X_{i,b}^a \frac{\partial}{\partial p_b}. \end{aligned}$$

We next impose the condition that the differentials be compatible, i.e.,

$$(5.6) \quad [d_C, d_F] = 0 \iff \{C, F\} = 0.$$

**5.1.1. Proposition.** *Condition (5.6) implies that  $(\mathfrak{a}, \mathfrak{a}^*, \mathfrak{a} \oplus \mathfrak{a}^*)$  is a Manin triple [12], with the Lie bracket on  $\mathfrak{a} \oplus \mathfrak{a}^*$  given by*

$$(5.7) \quad [e_i, e_j] = C_{ij}^k e_k, \quad [e^i, e^j] = F_k^{ij} e^k, \quad [e_i, e^j] = C_{ik}^j e^k + F_i^{jk} e_k.$$

where  $e_i$  and  $e^i$  are dual bases in  $\mathfrak{a}$  and  $\mathfrak{a}^*$  respectively. Equivalently,  $\mathfrak{a}$  is a Lie bialgebra. Moreover,  $\mathcal{F}_{\mathcal{X}}$  is a module over the Lie algebra  $\mathfrak{a} \oplus \mathfrak{a}^*$ .

The proof is straightforward.

That  $\mathcal{F}_{\mathcal{X}}$  is a module over  $\mathfrak{a} \oplus \mathfrak{a}^*$  means that under the mapping  $e_i \mapsto X_i$ ,  $e^i \mapsto X^i$ , the following commutation relations between vector fields are satisfied:

$$(5.8) \quad [X_i, X_j] = C_{ij}^k X_k, \quad [X^i, X^j] = F_k^{ij} X^k, \quad [X_i, X^j] = C_{ik}^j X^k + F_i^{jk} X_k.$$

It also follows from (5.6) that

$$(5.9) \quad X^{ia} X_i^b + X^{ib} X_i^a = 0.$$

**5.1.2. Zero locus reduction on a bi-QP manifold.** We next consider the submanifolds of the zero loci,  $\mathcal{L}_C \subset \mathcal{Z}_C$  and  $\mathcal{L}_F \subset \mathcal{Z}_F$  defined by  $(\theta^i = 0, p_a = 0)$  and  $(\xi_i = 0, p_a = 0)$ , respectively. Since  $\mathcal{L}_C$  and  $\mathcal{L}_F$  are coisotropic, we can apply Theorem 2.2.5. We thus have the respective antibrackets

$$(5.10) \quad \begin{aligned} \{\xi_i, \xi_j\}_C &= C_{ij}^k \xi_k, & \{\xi_i, q^a\}_C &= X_i^a, \\ \{\theta^i, \theta^j\}_F &= F_k^{ij} \theta^k, & \{\theta^i, q^a\}_F &= X^{ia}, \end{aligned}$$

on  $\mathcal{L}_C$  and  $\mathcal{L}_F$ .

**5.1.3. Proposition.** *The differential  $d_C$  induces a well-defined operator (vector field)  $\bar{d}_C = d_C|_{\mathcal{L}_F} : \mathcal{F}_{\mathcal{L}_F} \rightarrow \mathcal{F}_{\mathcal{L}_F}$  and the differential  $d_F$  induces an operator  $\bar{d}_F = d_F|_{\mathcal{L}_C} : \mathcal{F}_{\mathcal{L}_C} \rightarrow \mathcal{F}_{\mathcal{L}_C}$ . Thus,  $\mathcal{F}_{\mathcal{L}_F}$  ( $\mathcal{F}_{\mathcal{L}_C}$ ) is an odd differential Poisson algebra and  $\mathcal{L}_F$  (respectively,  $\mathcal{L}_C$ ) is an odd QP manifold.*

Thus, the manifolds  $\mathcal{L}_C$  and  $\mathcal{L}_F$  are equipped with  $\mathbf{Q}$  structures. We now proceed to the next step of the zero locus reduction.

Recall that the submanifold  $\mathcal{X} = \mathcal{L}_C \cap \mathcal{L}_F$  is determined by the equations  $p_a = \xi_i = \theta^j = 0$ . It is easy to see that  $\mathcal{X}$  is a coisotropic submanifold of  $\mathcal{L}_C$  and also a coisotropic submanifold of  $\mathcal{L}_F$ . On  $\mathcal{X}$ , we then have the Poisson bracket

$$(5.11) \quad \{\cdot, \cdot\}_{\mathcal{X}} = \{\cdot, \bar{d}_F \cdot\}_C = \{\cdot, \bar{d}_C \cdot\}_F$$

or in the coordinate form,

$$(5.12) \quad \{q^a, q^b\}_{\mathcal{X}} = X_i^a X^{ib}.$$

It follows from (5.9) that bracket (5.12) is skew-symmetric; the Jacobi identity follows from the compatibility of  $d_C$  and  $d_F$ .

**5.1.4. Coboundary bialgebras.** Up to this point, the situation was symmetric with respect to  $\theta^i \leftrightarrow \xi_i$ , but now we try to solve Eq. (5.6) for  $F$ . Namely, suppose that  $F$  is a coboundary

$$(5.13) \quad F = d_C r = \{C, r\},$$

where  $r = r^{ij} \xi_i \xi_j$  and  $r^{ij}$  is taken to be a constant matrix. Then the condition  $d_F^2 = 0$  yields

$$(5.14) \quad \{C, \{r, \{C, r\}\}\} = d_C \{r, d_C r\} = 0.$$

This is the generalized CYBE. An even stronger condition

$$(5.15) \quad \{r, d_C r\} = \{r, r\}_C = 0$$

leads to the CYBE (see (4.4)).

**5.2. Two differentials from a Lie algebra action.** We now look at the bicomplex setting from a somewhat different point of view. Rather than associating a second differential with a coalgebra structure, we construct a pair of differentials for a single Lie algebra. This subject attracts one's attention because of its possibly deep relation to the extended BRST symmetry [8, 9]. We now show that the bicomplex generalization of the zero locus reduction method induces the non-Abelian triplectic antibrackets on the space of common zeroes of the differentials.<sup>6</sup>

**5.2.1. Left and right  $\mathfrak{a}$  actions.** We consider the left and the right actions of  $\mathfrak{a}$  on  $\mathcal{X}$ . To illustrate the idea, we restrict ourselves to the case where  $\mathcal{X} = \mathcal{G}$  is the Lie group corresponding to the Lie algebra  $\mathfrak{a}$ . Let the basis elements  $e_i$  of  $\mathfrak{a}$  act on  $\mathcal{G}$  via the left invariant vector fields  $L_i$  (which correspond to the right action) and via the right invariant vector fields  $R_i$  (which correspond to the left action). Obviously,  $[L_i, R_j] = 0$ . Let  $q^a$  and  $p_a$  be the standard coordinates on  $T^*\mathcal{G}$ . Unlike in the case considered above, we introduce the doubled set of variables  $\xi_i^1, \xi_j^2, \theta_1^k$ , and  $\theta_2^l$ ,  $i, j, k, l = 1, \dots, \dim \mathfrak{a}$ , with the basic Poisson brackets

$$(5.16) \quad \{q^a, p_b\} = \delta_b^a, \quad \{\theta_1^i, \xi_j^1\} = \delta_j^i, \quad \{\theta_2^i, \xi_j^2\} = \delta_j^i.$$

The functions

$$(5.17) \quad \begin{aligned} \Omega^1 &= -p_a R_i^a \theta_1^i - \frac{1}{2} \xi_k^1 C_{ij}^k \theta_1^i \theta_1^j, \\ \Omega^2 &= -p_a L_i^a \theta_2^i - \frac{1}{2} \xi_k^2 C_{ij}^k \theta_2^i \theta_2^j \end{aligned}$$

satisfy  $\{\Omega^\alpha, \Omega^\beta\} = 0$  for  $\alpha, \beta = 1, 2$ , as follows immediately from the commutativity of the left- and right-invariant vector fields. These generating functions give rise to the anticommuting differentials  $\mathcal{Q}^a = \{\Omega^a, \cdot\}$ , thereby providing  $\mathcal{F}_{\text{ext}}$  with a bicomplex structure.

<sup>6</sup>The non-Abelian triplectic antibrackets were introduced in [13], see also [21], as the structure underlying a possible generalization of the well known Lagrangian version of the extended BRST quantization.

**5.2.2. Zero locus reduction in  $\mathcal{F}_{\text{ext}}$  and nonabelian triplectic antibrackets.** We now apply the zero locus reduction along the lines of Sec. 2. We identify the zero locus  $\mathcal{Z}_{Q^1}$  (respectively,  $\mathcal{Z}_{Q^2}$ ) of the differential  $Q^1$  (of  $Q^2$ ) determined by the equations  $\theta_1^i = 0$  and  $p_a = 0$  (respectively,  $\theta_2^i = 0$  and  $p_a = 0$ ). The intersection  $\mathcal{L} = \mathcal{Z}_{Q^1} \cap \mathcal{Z}_{Q^2}$  is then endowed with a pair of compatible antibrackets. Identifying  $\mathcal{F}_{\mathcal{L}}$  (functions on the intersection) with functions of  $q^a$ ,  $\xi_i^1$ , and  $\xi_j^2$ , we have

$$(5.18) \quad \begin{aligned} \{\xi_i^1, q^a\}_{Q^1} &= R_i^a, & \{\xi_i^1, \xi_j^1\}_{Q^1} &= C_{ij}^k \xi_k^1, \\ \{\xi_i^2, q^a\}_{Q^2} &= L_i^a, & \{\xi_i^2, \xi_j^2\}_{Q^2} &= C_{ij}^k \xi_k^2, \end{aligned}$$

with all the other brackets vanishing. These are precisely the non-Abelian triplectic antibrackets from [13].

## 6. CONCLUSIONS

Our results give a geometric interpretation to a number of structures involved in the BFV/BV formalism; the interpretation of the BRST cohomology in terms of the constraint surface geometry [22] can thus be extended in terms of geometry of a “more invariant” object—the zero locus  $\mathcal{Z}_Q$  that plays the role of the *extended constraint surface*. Although this is presently limited to the ghost number zero, it would be interesting to extend this interpretation to other ghost numbers. Another interesting application of the zero locus reduction consists in interpreting  $\mathcal{Z}_Q$  with the induced Poisson bracket in the BV formulation of a pure-gauge model as an extended phase space and the extended Poisson bracket in the BFV formulation of the same model [23]. As noted above, the zero locus reduction applies to finite-dimensional models; it would be interesting to extend it to local field theory, for example in the jet language formulation of the BRST formalism [24].

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## APPENDIX A. LIE ALGEBRA COHOMOLOGY AND THE (ANTI)BRACKET

Let  $\mathfrak{a}$  denote a Lie algebra of dimension  $N$  and  $\mathfrak{M}$  denote an  $\mathfrak{a}$ -module. We denote by

$$(A.1) \quad \bigwedge \mathfrak{a} = \bigoplus_{n=0}^N \wedge^n \mathfrak{a}$$

the exterior algebra of the vector space  $\mathfrak{a}$  and by  $\text{Sa}$  the symmetric tensor algebra.

The cohomology complex of  $\mathfrak{a}$  with coefficients in the module  $\mathfrak{M}$  is

$$(A.2) \quad C^*(\mathfrak{a}, \mathfrak{M}) = \{\text{Hom}(\bigwedge \mathfrak{a}, \mathfrak{M}), d\}.$$

Decomposition (A.1) induces the grading  $C^*(\mathfrak{a}, \mathfrak{M}) = \bigoplus_{n=0}^N C^n(\mathfrak{a}, \mathfrak{M})$ , where  $C^n(\mathfrak{a}, \mathfrak{M}) = \text{Hom}(\wedge^n \mathfrak{a}, \mathfrak{M})$ . The differential  $d$  has the degree 1 and acts as  $d : C^n(\mathfrak{a}, \mathfrak{M}) \rightarrow C^{n+1}(\mathfrak{a}, \mathfrak{M})$  via

$$(A.3) \quad (da)(g_1, \dots, g_{n+1}) = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j-1} a([g_i, g_j], g_1, \dots, \hat{g}_i, \dots, \hat{g}_j, \dots, g_{n+1}) + \\ + \sum_{1 \leq i \leq n+1} (-1)^i g_i a(g_1, \dots, \hat{g}_i, \dots, g_{n+1}), \quad a \in C^n(\mathfrak{a}, \mathfrak{M}).$$

We also use the simplified notation  $C^n = C^n(\mathfrak{a}, \mathfrak{M})$ .

We can identify the cohomology complex  $C^*(\mathfrak{a}, \mathfrak{M})$  with  $\bigwedge \mathfrak{a}^* \otimes \mathfrak{M}$  as follows.<sup>7</sup> Let  $e_i$  be a basis in  $\mathfrak{a}$ , with  $[e_i, e_j] = C_{ij}^k e_k$ . Let also  $\theta^i$  be the basis of  $\mathfrak{a}^*$  dual to  $e_i$ . The Grassmann algebra generated by  $\theta^i$  is then identified with  $\bigwedge \mathfrak{a}^*$ . To every cochain  $x \in \text{Hom}(\wedge^n \mathfrak{a}, \mathfrak{M})$ , we associate the element (with the summations implied)

$$(A.4) \quad \bar{x} = \frac{1}{n!} x(e_{i_1}, \dots, e_{i_n}) \theta^{i_1} \dots \theta^{i_n} \in \bigwedge \mathfrak{a}^* \otimes \mathfrak{M}.$$

The differential  $d$  then acts on  $\bigwedge \mathfrak{a}^* \otimes \mathfrak{M}$  as the differential operator

$$(A.5) \quad d = \frac{1}{2} C_{ij}^k \theta^i \theta^j \frac{\partial}{\partial \theta^k} - \theta^i X_i,$$

where  $X_i : \mathfrak{M} \rightarrow \mathfrak{M}$  is the action of  $e_i \in \mathfrak{a}$  on  $\mathfrak{M}$ .

We next specialize to the coefficients in  $\mathfrak{a}$  (viewed as the adjoint representation  $\mathfrak{a}$ -module). The complex is then endowed with the Gerstenhaber bracket [25],[26, and references therein]

$$\{ \cdot, \cdot \} : C^n \otimes C^m \rightarrow C^{n+m-1}$$

given by

$$(A.6) \quad \{x, y\} = x \circ y - (-1)^{(m+1)(n+1)} y \circ x, \quad x \in C^n, \quad y \in C^m$$

where

$$(A.7) \quad (x \circ y)(a_1, \dots, a_{n+m-1}) = \frac{1}{m!(n-1)!} \sum_{\sigma \in P_{n+m-1}} (-1)^\sigma x(a_{\sigma(1)}, \dots, a_{\sigma(n-1)}, y(a_{\sigma(n)}, \dots, a_{\sigma(n+m-1)}))$$

This makes  $\text{Hom}(\bigwedge \mathfrak{a}, \mathfrak{a})$  into a graded differential Lie algebra.

Let  $\xi_i$  denote the basis of  $\mathfrak{a}$  viewed as an  $\mathfrak{a}$ -module (equivalently, *coordinates* on  $\mathfrak{a}^*$ ). For each cochain  $x \in C^n$ , we then expand  $\bar{x}$  from (A.4) as

$$(A.8) \quad \bar{x} = \frac{1}{n!} \xi_j x^j(e_{i_1}, \dots, e_{i_n}) \theta^{i_1} \dots \theta^{i_n}$$

and rewrite the Gerstenhaber bracket as

$$(A.9) \quad \{\bar{x}, \bar{y}\} = \bar{x} \circ \bar{y} - (-1)^{(k+1)(l+1)} \bar{y} \circ \bar{x}, \quad \bar{x} \circ \bar{y} = \bar{x} \overleftarrow{\frac{\partial}{\partial \theta^i}} \frac{\partial}{\partial \xi_i} \bar{y}, \quad x \in C^k, \quad y \in C^l,$$

<sup>7</sup>We here assume that the algebras are finite dimensional or graded,  $\mathfrak{a} = \bigoplus_i \mathfrak{a}_i$ , with finite dimensional homogeneous spaces  $\mathfrak{a}_i$ , and  $\mathfrak{a}^*$  is by definition  $\mathfrak{a}^* = \bigoplus_i \mathfrak{a}_i^*$ , where  $\mathfrak{a}_i^*$  are finite dimensional spaces dual to  $\mathfrak{a}_i$ .

where  $\frac{\overleftarrow{\partial}}{\partial \theta^i}$  is the (right) derivative in the Grassmann algebra and the  $\frac{\partial}{\partial \xi_i}$  operation is defined on the elements of form (A.8) as the contraction with the element  $\xi_i^*$  of the dual basis in  $\mathfrak{a}^*$ . The differential then becomes

$$(A.10) \quad d = \{ -\frac{1}{2} C_{ij}^k \xi_k \theta^i \theta^j, \cdot \} = \frac{1}{2} C_{ij}^k \theta^i \theta^j \frac{\partial}{\partial \theta^k} - \xi_k \theta^i C_{ij}^k \frac{\partial}{\partial \xi_j}.$$

On the elements  $\underline{a}$  as in (A.8), the second term represents the adjoint action (in accordance with the above choice  $\mathfrak{M} = \mathfrak{a}$ ). Equation (A.9) suggests the interpretation of a Poisson/Batalin–Vilkovisky bracket. As it stands, however, (A.9) can be neither of these, since no associative supercommutative multiplication has been defined on the cochains.<sup>8</sup> There are two remarkable possibilities to embed  $C^*(\mathfrak{a}, \mathfrak{M}) = C^*(\mathfrak{a}, \mathfrak{a})$  into a complex endowed with a multiplication: the complex

$$(A.11) \quad C^*(\mathfrak{a}, S\mathfrak{a}) = \bigwedge \mathfrak{a}^* \otimes S\mathfrak{a}$$

corresponding to the BV quantization, or the complex

$$(A.12) \quad C^*(\mathfrak{a}, \bigwedge \mathfrak{a}) = \bigwedge \mathfrak{a}^* \otimes \bigwedge \mathfrak{a}$$

corresponding to the BFV quantization. Geometrically, these two possibilities correspond to even and odd QP manifolds (see Definition 2.1).

Choosing  $\mathfrak{M} = S\mathfrak{a}$ , we have the complex  $\bigoplus_{m,n} \text{Hom}(\bigwedge^m \mathfrak{a}, S^n \mathfrak{a})$ , which can be viewed as the associative supercommutative algebra generated by the variables  $\theta^i$  and  $\xi_j$  satisfying  $\xi_i \xi_j - \xi_j \xi_i = 0$ ,  $\theta^i \theta^j + \theta^j \theta^i = 0$ , and  $\theta^i \xi_j - \xi_j \theta^i = 0$ .<sup>9</sup> It then follows that (A.9) can be extended to an odd bracket on this complex. The differential extends to  $\text{Hom}(\bigwedge \mathfrak{a}, S\mathfrak{a})$  by the same formula  $d = \{C_0, \cdot\}$ ,  $C_0 = -\frac{1}{2} C_{ij}^k \xi_k \theta^i \theta^j$ . The complex is endowed with the grading known as the *ghost number* in the BV quantization or as the Weyl complex grading in homology theory: for a cochain  $x \in \text{Hom}(\bigwedge^m \mathfrak{a}, S^n \mathfrak{a})$ , one has  $\text{gh}(x) = m - 2n$ .

On the other hand, taking the coefficients to be the *exterior* algebra  $\bigwedge \mathfrak{a}$ , we can extend (A.9) to an even bracket. With  $\bigwedge \mathfrak{a}$  identified with the algebra generated by  $\xi_i$  viewed as *anticommuting variables* (with obvious modifications in the case where  $\mathfrak{a}$  is a Lie *superalgebra*, see footnote 9), the bracket becomes the Poisson bracket on the space  $\bigwedge \mathfrak{a}^* \otimes \bigwedge \mathfrak{a}$  (which is identified with functions of  $\theta^i$  and  $\xi_j$ ; we also assume that  $\xi_i \theta^j + \theta^j \xi_i = 0$  in addition to  $\xi_i \xi_j + \xi_j \xi_i = 0$ ). The ghost number grading on this complex taken from the BFV quantization is  $\text{gh}(x) = m - n$  for an element  $x \in \text{Hom}(\bigwedge^m \mathfrak{a}, \bigwedge^n \mathfrak{a})$ .

<sup>8</sup>Superficially, the bracket in (A.9) has the grade  $-1$  since it maps as  $C^m \times C^n \rightarrow C^{m+n-1}$ , however the gradings of all the terms in the complex can be shifted by 1, after which the bracket becomes a grade-0 operation. On the other hand, an associative graded commutative multiplication defined on the complex would fix the grading, and (A.9) would become either the Batalin–Vilkovisky or the Poisson bracket.

<sup>9</sup>These relations between  $\theta$  and  $\xi$  variables correspond to the case (tacitly implied in most of our formulae) where  $\mathfrak{a}$  is a Lie algebra, *not* a superalgebra; then the Grassmann parities are simply  $\mathfrak{p}(\xi_i) = 0$  and  $\mathfrak{p}(\theta^i) = 1$ . However, if  $\mathfrak{a}$  is a Lie superalgebra, let  $\mathfrak{p}(e_i) = \varepsilon_i$  be the Grassmann parities of its generators. Then  $\mathfrak{p}(\xi_i) = \varepsilon_i$  and  $\mathfrak{p}(\theta^i) = \varepsilon_i + 1$ , and therefore,  $\xi_i \xi_j - (-1)^{\varepsilon_i \varepsilon_j} \xi_j \xi_i = 0$ ,  $\theta^i \theta^j - (-1)^{(\varepsilon_i+1)(\varepsilon_j+1)} \theta^j \theta^i = 0$ , and  $\xi_i \theta^j - (-1)^{\varepsilon_i(\varepsilon_j+1)} \theta^j \xi_i = 0$ .

The coefficients can be further extended (cf. [26]) by  $\mathfrak{M} = \mathcal{F}_{\mathcal{M}}$ , the algebra of smooth functions on a manifold  $\mathcal{M}$  such that  $\mathfrak{a}$  acts on  $\mathcal{F}_{\mathcal{M}}$  by *derivations* (vector fields on  $\mathcal{M}$ ). We write  $X_i$  for the image of the basis elements of  $\mathfrak{a}$  in  $\text{Vect}_{\mathcal{M}}$ . In accordance with the BRST paradigm, one wishes the vector fields representing the action of  $\mathfrak{a}$  on  $\mathcal{M}$  to be Hamiltonian with respect to a bracket structure. For *even*  $\xi_i$ , this can be achieved by replacing  $\mathcal{M}$  with the odd cotangent bundle  $\Pi T^* \mathcal{M}$  and, thus, the algebra  $\mathcal{F}_{\mathcal{M}}$  with the algebra  $\mathcal{F}_{\Pi T^* \mathcal{M}}$  of smooth functions on the odd cotangent bundle. Then each vector field  $V = V^a \frac{\partial}{\partial q^a}$  on  $\mathcal{M}$  is generated by the canonical antibracket structure on  $\Pi T^* \mathcal{M}$ ; the action of the basis elements  $X_i = X_i^a \frac{\partial}{\partial q^a}$  on functions is given by the antibracket

$$(A.13) \quad X_i F = -\{X_i^a q_a^*, F\}, \quad F \in \mathcal{F}_{\mathcal{M}},$$

with  $q_a^*$  being the standard coordinates on the fibers of  $\Pi T^* \mathcal{M}$  (and the standard antibracket given by  $\{q^a, q_b^*\} = \delta_b^a$ ).

For odd  $\xi_i$ , similarly, we can consider the functions  $\mathcal{F}_{T^* \mathcal{M}}$  on the cotangent bundle, which allows the action of  $\mathfrak{a}$  to be implemented by the bracket on  $\mathcal{F}_{T^* \mathcal{M}}$  (the same formula (A.13) for the bracket, where now  $q_a^*$  are the canonical coordinates on the fibers of  $T^* \mathcal{M}$ ).

We note, however, that the differential

$$(A.14) \quad d = \{ -\frac{1}{2} C_{ij}^k \xi_k \theta^i \theta^j, \cdot \} + \theta^i \{ X_i^a q_a^*, \cdot \}$$

in either of the complexes

$$(A.15) \quad \mathbb{C}_{\text{odd}}^*(\mathfrak{a}, \mathcal{M}) = \mathbb{C}^*(\mathfrak{a}, \mathbf{S}\mathfrak{a}) \otimes \mathcal{F}_{\Pi T^* \mathcal{M}},$$

$$(A.16) \quad \mathbb{C}_{\text{even}}^*(\mathfrak{a}, \mathcal{M}) = \mathbb{C}^*(\mathfrak{a}, \bigwedge \mathfrak{a}) \otimes \mathcal{F}_{T^* \mathcal{M}}$$

is *not* compatible with the bracket. Remarkably, the compatibility can be achieved by changing the differentials such that (A.15) and (A.16) become the well-known BV and BFV complexes used in the Lagrangian and Hamiltonian quantization of gauge theories. The term to be added to the differential is the Koszul differential involving precisely the same “auxiliary” variables  $\xi_i$  that were originally introduced to rewrite the Gerstenhaber bracket in the “geometric” form.

To conclude, we note that we have given a homological interpretation of the structures appearing in the BRST quantization in the example of a Lie algebra structure (i.e., in the case where the constraints or gauge generators form a Lie algebra). In the most general setting, the BRST charge and the master action in the BFV and BV cases, respectively, can be considered as the generating functions for the  $L_{\infty}$  algebras [7] (see also [4]). From this general standpoint, the Lie algebra structure appears as a particular case.

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